Appendix 6A  Noise Figure for Two-Port Network

Ludwig, RF Circuit Design: Theory & Applications, Appendix H

\[ R_S \]

\[ R_L = R_S \]

\[ V_n \]

Figure H-1  Noise voltage of a circuit.

According to this circuit, the noise power is treated as if a noise voltage source drives a noiseless resistor \( R_S \). Under matching condition \( R_S = R_L \), the noise power of the resistor is given as

\[
P_n = \frac{V_{n\text{RMS}}^2}{4R_S} = kTB
\]

(H.4)

from which the RMS noise voltage is found

\[
V_{n\text{RMS}} = \sqrt{4kTB}R_S
\]

(H.5)

To keep the notation simple (and since no ambiguity will arise) the subscript RMS is dropped (i.e., \( V_{n\text{RMS}} \equiv V_n \)). In general, we represent a noisy resistor \( R \) as a noise voltage source in series with the noise free resistor \( R \) (Thévenin equivalent circuit) or as a noise current source \( I_n = \sqrt{4kTB}/R \) in shunt with a noise free resistor, as shown in Figure H-2.

![Figure H-2](image_url)

Figure H-2  Equivalent voltage and current models for noisy resistor.

If the the bandwidth is eliminated from (H.5) we can define a so-called spectral noise voltage and a spectral noise current:

\[
\overline{V}_n = \frac{V_n}{\sqrt{B}} \quad \text{and} \quad \overline{I}_n = \frac{I_n}{\sqrt{B}}
\]

(H.6)

whose units are given in \( \text{V}/\sqrt{\text{Hz}} \) and \( \text{A}/\sqrt{\text{Hz}} \).
H.2 Noisy Two-Port Networks

The previous analysis can be expanded to two-port networks. Figure H-3 shows a noisy network and the equivalent noise-free network augmented by two current noise sources \( I_{n1} \) and \( I_{n2} \).

![Fig H-3 Noisy two-port network and its equivalent representation.](image)

In \( Y \)-parameter matrix representation we can write

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} + \begin{bmatrix}
I_{n1} \\
I_{n2}
\end{bmatrix} \quad (H.11)
\]

A more useful representation is obtained when rearranging (H.11) as follows:

\[
V_1 = -\frac{Y_{22}}{Y_{21}}V_2 + \frac{1}{Y_{21}}I_2 - \frac{1}{Y_{21}}I_{n2} \quad (H.12a)
\]

and

\[
I_1 = \frac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_{21}}V_2 + \frac{Y_{11}}{Y_{21}}I_2 + I_{n1} - \frac{Y_{11}}{Y_{21}}I_{n2} \quad (H.12b)
\]

* Define transformed voltage source \( V_n = -I_{n2}/Y_{21} \)

* Define transformed current source \( I_n = I_{n1} - (Y_{11}/Y_{21})I_{n2} \)

\[\Rightarrow\] Fig H-4

![Fig H-4 Transformed network model with noise sources at the input.](image)

\[ (H.12b) \quad I_1 = V_2(Y_{11}Y_{22} - Y_{12}Y_{21})/Y_{21} + [(Y_{11}/Y_{21})I_2] + [I_{n1} - (Y_{11}/Y_{21})I_{n2}] \\
= V_2(Y_{11}Y_{22} - Y_{12}Y_{21})/Y_{21} + [-V_n(I_2Y_{11}/I_{n2})] + [I_n] \quad (?) \]
Example H-1: Noise analysis of a low-frequency BJT amplifier

In Figure H-5 a simplified BJT amplifier is treated as a two-port network consisting of the following parameters: \( V_S = 25 \text{ mV} \), \( R_S \approx 50 \Omega \), \( R_{in} = 200 \Omega \), voltage gain \( g_v = 50 \), and measurement bandwidth \( B = 1 \text{ MHz} \). The spectral noise voltage and current of the amplifier are given by the manufacturer as \( V_n = 9 \text{ nV} / \sqrt{\text{Hz}} \) and \( I_n = 9 \text{ fA} / \sqrt{\text{Hz}} \). Find the signal-to-noise ratio \( \text{SNR} = 20 \log(V_2/V_{n2}) \) at the output.

![Amplifier model and network representation with noise sources.](image)

The output voltage \( V_2 \) is directly found from \( V_2 = g_v R_{in}/(R_{in} + R_S) V_S = 1 \text{ V} \). The spectral noise sources of the network are next expressed in RMS noise voltage and current:

\[
V_n = \sqrt{V_n^2 \cdot B} = 9 \mu\text{V} \quad \text{and} \quad I_n = \sqrt{I_n^2 \cdot B} = 9 \text{ pA}
\]

The voltage source creates through the voltage divider rule the following noise voltage across \( R_{in} \):

\[
\frac{R_{in}}{R_{in} + R_S} V_n = 7.2 \text{ nV}
\]

The noise current source is responsible for the noise voltage of

\[
\frac{R_{in} R_S}{R_{in} + R_S} I_n = 0.36 \text{ nV}
\]

Finally, the source resistor contributes the voltage

\[
\frac{R_{in}}{R_{in} + R_S} V_{ns} = 728 \text{ nV}
\]

where \( V_{ns} = \sqrt{4kTBR_S} = 910 \text{ nV} \), assuming \( T = 300^\circ\text{K} \).

Therefore, the total noise voltage at the output is

\[
V_{n2} = g_v \left( \frac{R_{in}}{R_{in} + R_S} V_n \right)^2 + \left( \frac{R_{in} R_S}{R_{in} + R_S} I_n \right)^2 + \left( \frac{R_{in}}{R_{in} + R_S} V_{ns} \right)^2 = 36.4 \mu\text{V}
\]

Finally, the signal to noise ratio is

\[
\text{SNR} = 20 \log \left( \frac{V_2}{V_{n2}} \right) = 122.8 \text{ dB}
\]
H.3 Noise Figure for Two-Port Network

The noise figure is defined as the ratio between the SNR at the input to the SNR at the output port of a network. Specifically, Figure H-6 depicts the relevant power flow conventions, including the noise representation of the source $Z_S$.

![Diagram](image)

**Figure H-6** Generic noise model for noise figure computation.

The noise figure $F$ can be cast into several equivalent representations. The first form involves the ratios of the signal to noise power at the input and output ports:

$$F = \frac{P_1/P_n}{P_2/P_n} = \frac{P_{n2}/P_2}{P_{n1}/P_1}$$

\[ (H.14) \]

**Signal power**

$$P_1 = \frac{1}{2} \Re \left\{ \frac{Z_{in}}{Z_S + Z_{in}} \right\} V_s^2 = \frac{V_s^2}{2} \left( \frac{R_{in}}{R_{in} + R_s} \right)^2 + \left( \frac{X_{in} + X_s}{Z_S + Z_{in}} \right)^2$$

**Thermal noise power**

$$P_{n1} = 4kTR_s B \left( \frac{\Re \left\{ Z_{in} \right\}}{|Z_S + Z_{in}|^2} \right) V_{ns}^2 = V_{ns}^2 \left( \frac{\Re \left\{ Z_{in} \right\}}{|Z_S + Z_{in}|^2} \right)$$

$$\Rightarrow P_1 / P_{n1} = \frac{V_s^2}{V_{ns}^2}$$

**Note**

$$P_{n2} = G_A P_{n1} + P_{ni}$$

$$\Rightarrow P_{ni} \text{ (internally generated noise) takes into account the noise sources associated with } V_n \text{ & } I_n$$

$$\Rightarrow V_{ns}^2 \text{ replaced by all noise sources } [V_{ns}^2 + V_n^2 + (I_n R_{in})^2]$$

$$\Rightarrow P_2 / P_{n2} = \frac{V_s^2}{|V_{ns}^2 + V_n^2 + (I_n R_{in})^2|}$$

$$\Rightarrow \text{Noise figure } F = \frac{P_{n2}/P_2}{P_{n1}/P_1} = \frac{V_{ns}^2 + V_n^2 + (I_n R_{in})^2}{V_{ns}^2 (4kT R_{in} B)} = V_n^2 + (I_n R_{in})^2 \frac{4kT R_{in}}{4kT R_{in} B}$$
The preceding treatment does not take into account the fact that the same noise mechanisms are usually responsible for both $V_n$ and $I_n$. Thus, these sources are, to a certain degree, correlated. This can be incorporated into the noise model by splitting $I_n$ into an uncorrelated, $I_{nu}$, and a correlated current, $I_{nc}$, contribution, respectively. The correlated current contribution is related to the noise voltage $V_n$ via a complex correlation factor $Y_C = G_C + jB_C$, such that $I_{nc} = Y_C V_n$. Since it is more convenient to deal with noise currents than voltages for our network, we convert the source into an equivalent Norton representation, as seen in Figure H-7.

![Figure H-7](image)

**Figure H-7** Noise sources modeled at network input.

The total RMS noise current $I_{n\text{tot}}$ under short circuit input conditions can be expressed as

$$I_{n\text{tot}}^2 = I_{ns}^2 + V_n^2(Y_S + Y_C)^2 + I_{nu}^2$$  \hspace{1cm} (H.22)

where $I_{nc} = Y_C V_n$ and $I_n = V_n Y_S$ are combined because of their correlation. We can now rewrite (H.21) as

$$F = \frac{I_{ns}^2 + V_n^2(Y_S + Y_C)^2 + I_{nu}^2}{I_{ns}^2}$$  \hspace{1cm} (H.23)

Under the assumption that all noise sources are represented by an equivalent thermal noise source, we identify in (H.23)

<table>
<thead>
<tr>
<th>$I_{ns}^2 = 4kTBG_S$</th>
<th>noise due to the source $Y_S = G_S + jB_S$</th>
<th>(H.24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{nu}^2 = 4kTBG_u$</td>
<td>noise due to the equivalent noise conductance $G_u$</td>
<td>(H.25)</td>
</tr>
<tr>
<td>$V_n^2 = 4kTBR_n$</td>
<td>noise due to the equivalent noise resistance $R_n$</td>
<td>(H.26)</td>
</tr>
</tbody>
</table>

Inserting (H.24)–(H.26) into (H.23) gives

$$F = 1 + \frac{G_u + R_n}{G_S} \frac{|Y_S + Y_C|^2}{G_S} = 1 + \frac{G_u}{G_S} \left(\frac{R_n}{G_S} \left[(G_S + G_C)^2 + (B_S + B_C)^2\right]\right)$$  \hspace{1cm} (H.27)

**To minimize noise**

$$F = 1 + \frac{G_u}{G_S} \frac{R_n}{G_S} \left[(G_S + G_C)^2 + (B_S + B_C)^2\right]$$

$\Rightarrow$ let $B_S = -B_C \Rightarrow B_S + B_C = 0 \Rightarrow F = 1 + \frac{G_u}{G_S} \frac{R_n}{G_S} \left[(G_S + G_C)^2\right]$. 
Next, the remaining expression is minimized with respect to $G_S$; that is,

$$
\frac{dF(B_S = -B_C)}{dG_S} = \frac{1}{G^2_{\text{Sopt}}} \{ R_n [2 G_{\text{Sopt}} (G_{\text{Sopt}} + G_C) - (G_{\text{Sopt}} + G_C)^2] \} = 0 \quad (H.28)
$$

which yields the explicit optimum value

$$
G_{\text{Sopt}} = \frac{1}{\sqrt{R_n}} \frac{R_n G^2_C + G_u}{R_n}
$$

(H.29)

The **minimum noise figure** is thus obtained by the optimal source admittance

$$
Y_{\text{Sopt}} = \frac{1}{\sqrt{R_n}} \left( \frac{R_n G^2_C + G_u}{R_n} \right) - j B_C
$$

(H.30)

Substituting (H.29) into (H.27) results in the expression

$$
F_{\text{min}} = 1 + \frac{G_u}{G_{\text{Sopt}}} + \frac{R_n}{G_{\text{Sopt}}} (G_{\text{Sopt}} + G_C)^2
$$

(H.31)

Eliminating $G_u$ in (H.31) by using $G_u = R_n G^2_{\text{Sopt}} - R_n G^2_C$ from (H.29) gives

$$
F_{\text{min}} = 1 + 2 R_n (G_{\text{Sopt}} + G_C)
$$

(H.32)

⇒ $F_{\text{min}}$ is typically provided by the device manufacturer

⇒ dependent on **frequency & bias conditions**

⇒ (H.32) incorporated into (H.27)

$$
F = F_{\text{min}} - 2 R_n G_{\text{Sopt}} - 2 R_n G_C + \frac{G_u}{G_{\text{Sopt}}} + \frac{R_n}{G_{\text{Sopt}}} [(G_S + G_C)^2 + (B_S - B_{\text{Sopt}})^2]
$$

(H.33)

⇒ Replacing $G_u$ by $G_u = R_n G^2_{\text{Sopt}} - R_n G^2_C$

$$
F = F_{\text{min}} + \frac{R_n}{G_{\text{Sopt}}} [(G_S - G_{\text{Sopt}})^2 + (B_S - B_{\text{Sopt}})^2] = F_{\text{min}} + \frac{R_n}{G_{\text{Sopt}}} |Y_S - Y_{\text{Sopt}}|^2
$$

(H.34)

This is the starting point of our noise circle analysis in Section 9.5. Based on the characteristic line impedance $Z_0 = 1/Y_0$, (H.34) is often expressed in terms of normalized noise resistance $r_n = R_n/Z_0$, conductance $g_S = G_S/Y_0$, and admittances $\gamma_S = Y_S/Y_0$, $\gamma_{\text{Sopt}} = Y_{\text{Sopt}}/Y_0$ in the form

$$
F = F_{\text{min}} + \frac{r_n}{g_S} |\gamma_S - \gamma_{\text{Sopt}}|^2
$$

(H.35)

$\gamma_S = g_S + j b_S = \text{source admittance}$ presented to transistor

$\gamma_{\text{Sopt}} = \text{optimum source admittance}$ that results in **minimum noise figure**

$F_{\text{min}} = \text{minimum noise figure}$ of transistor, attained when $Y_S = Y_{\text{opt}}$

$r_n = \text{equivalent noise resistance}$ of transistor

$g_S = \text{real part of source admittance}$